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Some Infinite Frobenius Groups

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A classical result in the theory of finite permutation groups, due to Frobenius, states that a transitive permutation group in which only the identity fixes two or more points has a regular normal subgroup. It is then a well known result of Thompson that such a regular normal subgroup is soluble, and hence nilpotent. It is reasonable to ask to what extent analogues may exist for infinite groups, both with regard to the existence of regular normal subgroups and to their nilpotence; questions of this nature seem to appear in the consideration of the relationship between infinite permutation groups and model theory. Since the only known proof of Frobenius' theorem in the finite case depends on character theory, one should expect any results in the infinite case to be derived from the finite result. Indeed, the analogous statement is not true for all infinite groups.

An example is given by Kegel and Wehrfritz [3; p. 51]. If G is the free group on two generators x and y, then G may be represented faithfully as a Frobenius group on the cosets of the subgroup $H = \langle [x, y] \rangle$ and H does not even have a proper normal supplement.

For locally finite groups, the expected analogue does hold. Kegel and Wehrfritz prove that a locally finite Frobenius group G has a regular normal subgroup N; indeed, they show also that N is then nilpotent. This is so because an element of prime order p which has a fixed point acts fixed-point-freely on N by conjugation so that every finitely generated subgroup of N is nilpotent of class at most $k(p)$ where $k(p)$ is the bound given by a theorem of Higman [2]. But then all commutators in N of suitable weight are trivial.

In this paper, we use the term *Frobenius group* to describe a transitive permutation group in which some nonidentity element fixes a point, but none fixes two or more points. By a *Frobenius kernel*, in contrast to [3], we shall always mean a *regular* normal subgroup in a Frobenius group, if such should exist; in this case a Frobenius group G is the semidirect

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product HN of a point stabiliser H and the Frobenius kernel N. There are thus two immediate questions that arise: the first is that of existence, and the second is that of nilpotence. In fact, it is easy to see that a Frobenius kernel can exist without necessarily being nilpotent. Let X be an arbitrary simple group and form the natural semidirect product G of the (restricted) direct product $N=\prod_{i=-\infty}^{\infty} X_i$ where X_i is isomorphic to X for all i by an infinite cyclic group H acting naturally on the direct product. Then $C_N(h)=1$ for all $h\in H^\#$ so that G acts as a Frobenius group on the cosets of H. However, we can ask about Frobenius groups where the Frobenius kernel is "almost" nilpotent, and this will pose some questions about Burnside groups and their automorphisms which we shall not pursue here.

Our main result will be to examine Frobenius groups under a hypothesis slightly weaker than local finiteness, though containing that as a special case, and to obtain some information about the existence and structure of a Frobenius kernel under further assumptions.

THEOREM. *Let G be a Frobenius group of infinite degree, acting on a set Ω. Suppose that any two fixed-point-free elements of G generate a finite subgroup. Then the set of fixed-point-free elements of G, together with the identity, forms a normal subgroup N. The subgroup N acts transitively, and thus regularly, on Ω if either of the two following conditions holds:*

(i) *any two elements of G generate a finite subgroup, or*

(ii) *$N\neq 1$ and G acts doubly transitively on Ω.*

If (ii) *holds, then N is abelian. If* (i) *is replaced by*

(i′) *any three elements of G generate a finite subgroup,*

then N satisfies an Engel condition.

Before proving this result, it is perhaps worth commenting that some conditions are necessary to ensure that N is a Frobenius kernel, so that this is a real question too, even given the existence of the subgroup. In the finite case, transitivity arises almost by accident. Counting the number of elements of a Frobenius group which fix some point leaves exactly $n-1$ fixed-point-free elements, where n is the degree. However, it is possible in the infinite situation even to have *no* fixed-point-free elements. A group G of prime exponent has been constructed by E. Rips (unpublished, but see [6]) in which every nonidentity proper subgroup has prime order and in which all such subgroups are conjugate; then G acts as a Frobenius group by conjugation on these subgroups, with every nonidentity element fixing the subgroup which contains it, and no other. Worse still, G is simple! This group is not doubly transitive, but we note, with respect to condition (ii), that one question of current interest in model theory is whether there exist doubly transitive infinite Frobenius groups without fixed-point-free elements.

Proof of the Theorem. We observe that the existence of the normal subgroup N is simply the assertion that the product of two fixed-point-free elements is either fixed-point-free also or else the identity. Suppose that g_1 and g_2 are two fixed-point-free elements whose product $g_1 g_2$ is a nonidentity element with a fixed point ω. Consider the action of the subgroup $L = \langle g_1, g_2 \rangle$ on Ω and let Δ denote the orbit of L containing ω. By hypothesis, L is finite, as then is Δ. Now L acts as a Frobenius group on Δ, so that $g_1 g_2$ cannot in fact have a fixed point in Δ. Hence the fixed-point-free elements together with the identity form a normal subgroup.

Suppose now that (i) holds. Let $\omega_1, \omega_2 \in \Omega$ and let $g \in G$ with $\omega_1 g = \omega_2$. If $\omega_1 = \omega_2$ or $g \in N$, we are done. Otherwise, suppose that g fixes a point $\omega_3 \in \Omega$. Pick $h \in G$ with $\omega_1 h = \omega_3$ and put $H = \langle g, h \rangle$. By assumption, H is finite. Let Γ be the orbit of H containing ω_3. Then Γ contains ω_1 and ω_2 also. H acts on Γ as a Frobenius group and the Frobenius kernel K of H contains an element k such that $\omega_1 k = \omega_2$. Now H must act on its other orbits on Ω either as a Frobenius group or else regularly; since $K = F(H)$, the Fitting subgroup of H, any action as a faithful Frobenius group is uniquely determined. It follows that the action is in fact regular, so that $k \in N$.

Suppose that (ii) holds. By hypothesis, $N \neq 1$. Fix $\omega \in \Omega$, let Δ be the orbit of N containing ω, and let $\omega' \in \Delta - \{\omega\}$. For $g \in G_\omega$, since $\omega' = \omega n$ for some $n \in N$, we have $\omega' = \omega g^{-1} n$. Then $\omega' g = \omega g^{-1} n g \in \Delta$ and $\Delta \supseteq \Omega - \{\omega\}$ since G is doubly transitive. Thus N is transitive on Ω.

We now consider the possible structure of N. Let $H = G_\omega$ be a point stabiliser. Exactly as in the finite case, H and N contain the centralisers of their nonidentity elements. (See, for example, and especially in view of our dedication, [1].)

If (ii) holds, then $G = HN$ since N is a Frobenius kernel, and then H acts transitively on the nonidentity elements of N by conjugation; hence every nonidentity element of N has the same prime order p. Suppose that p is odd. Then an element x of H which inverts a nonidentity element of N is an involution and $C_N(x) = 1$. If $n \in N$, then $\langle n, n^x \rangle$ is finite and $x \in N(\langle n, n^x \rangle)$, whence x inverts every element of N. Thus N is abelian by Neumann's extension [4] of Burnside's result since every element of N has a unique square root. If $p = 2$, then N has exponent 2; hence in either case N is abelian.

If (i′) holds, then, for $n_1, n_2 \in N$ and $h \in H^\#$, the subgroup $\langle n_1, n_2, h \rangle$ is a finite group acting as a Frobenius group on the orbit containing ω, with n_1 and n_2 in the Frobenius kernel. But we may now choose h of fixed prime order p as n_1 and n_2 vary. Thus the subgroup N satisfies the $k(p)$th Engel condition, where $k(p)$ is the bound given by Higman's theorem.

We remark that the argument used in this final step can be modified to manufacture further results. For example, if we were to assume that G contained non-fixed-point-free elements of order at most a given prime p (for example, by specifying existence, or by bounding the exponent of the whole group), then the further assumption that every $k(p)+1$ generator subgroup of G was finite would enable us to deduce that all commutators in N of weight $k(p)$ were trivial, whence N would be nilpotent.

We shall complete this note with an example of a Frobenius group which has an insoluble Frobenius kernel, every finitely generated subgroup of which is even nilpotent.

The Burnside problem has an affirmative answer for exponent 4; namely, every finitely generated group of exponent 4 is finite. Let N be the free group of countable rank in the variety of groups of exponent 4. By a result of Razmyslov [5], N is not soluble yet every finitely generated subgroup of N is finite and hence nilpotent. Let F be the free group on generators $\{x_i \mid i \in \mathbf{Z}\}$. Then N is a homomorphic image of F and the images $\{\bar{x}_i \mid i \in \mathbf{Z}\}$ generate N.

The automorphism $\alpha: x_i \to x_{i+1}$ of F induces an automorphism of N. If α had a nontrivial fixed point y in N, then y would lie in some subgroup $\bar{A}$ and we may suppose that $\bar{A} = \langle \bar{x}_1, ..., \bar{x}_r \rangle$. Now $\bar{A}$ is isomorphic to the Burnside group $B(4, r)$ since $B(4, r)$ is the homomorphic image of F obtained by mapping $x_1, ..., x_r$ to a set of generators of $B(4, r)$ and x_i to 1 if $i \neq 1, ..., r$. Now $y\alpha^r = y$ so that $y \in \bar{B} = \langle \bar{x}_{r+1}, ..., \bar{x}_{2r} \rangle$. However, N has a homomorphic image of the form $\bar{A} \times \bar{B}$ obtained from F by putting $x_k = 1$ for $k \neq 1, ..., 2r$, and $[x_i, x_j] = 1$ for $i = 1, ..., r$ and $j = r+1, ..., 2r$ in addition to the relations which define N. Hence $\bar{A} \cap \bar{B} = 1$. So α can fix only the identity. Now we may form a Frobenius group by taking the semidirect product of N by an infinite cyclic group H with conjugation given by the automorphism α, and N is the desired Frobenius kernel.

We note that this example has a locally finite 2-group for the regular normal subgroup. In line with our theorem, it would be nice to have a finiteness condition only on 2-generator subgroups. The construction of a Frobenius group as above can be repeated for any Burnside group on a countable number of generators; for example, if it should be the case that a 2-generator group of exponent 5 is finite, though not for some larger number of generators, then one would have a more interesting example!

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